Bachelor of Science (B.Sc.) Semester—II (C.B.S.) Examination

MATHEMATICS

Compulsory Paper—II

(M_4—Vector Calculus and Improper Integrals)

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the ***five*** questions.

(2) All questions carry equal marks.

(3) Question Nos. 1 to 4 have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2 \cos 3t$, $z = 2 \sin 3t$, where t is time.

(i) Determine its velocity and acceleration at any time.

(ii) Find the magnitudes of the velocity and acceleration at $t = 0$. 6

(B) Prove $\nabla^2 r^n = n(n+1)\, r^{n-2}$, where n is a constant and $r^2 = x^2 + y^2 + z^2$. 6

OR

1. (C) Prove : (a) $\overline{\nabla}\times(\overline{\nabla}\phi)=0$, (b) $\overline{\nabla}\, o\, (\overline{\nabla}\times\overline{A}) = 0$

where $\phi = \phi(x, y, z)$ and $\overline{A}=A_1\overline{i}+A_2\overline{j}+A_3\overline{k}$ are differentiable scalar point function and vector point function of x, y, z. 6

(D) If $\overline{A} = (2y + 3)\ \overline{i}\ + x\, z\ \overline{j} + (yz - x)\ \overline{k}$, then evaluate $\int_C \overline{A}\, o\, d\overline{r}$ along the path C : $x = 2t^2$, $y = t$, $z = t^3$ from $t = 0$ to $t = 1$. 6

UNIT—II

2. (A) Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = 1$. 6

(B) Change the order of integration :

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dxdy.$$ 6

OR

2. (C) Evaluate $\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$ by changing to polar coordinates. 6

(D) Evaluate :

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz. \qquad 6$$

UNIT—III

3. (A) Evaluate $\iiint_V (2x + y)dv$, where v is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$. 6

(B) Evaluate $\oint_C (x^2 + y^2)\,dx + 3xy^2dy$ by using Green's theorem in a plane, where C is a circle of radius 2 with centre at the origin of the xy-plane, traversed in the positive sense. 6

OR

3. (C) Evaluate $\iint_S \bar{F} \cdot \bar{n}\,ds$ by using the divergence theorem, where $\bar{F} = 2xy\,\bar{i} + yz^2\bar{j} + xz\bar{k}$ and S is the surface of the region bounded by $x = 0$, $y = 0$, $y = 3$, $z = 0$ and $x + 2z = 6$. 6

(D) Verify Stoke's theorem for :

$\vec{A} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. 6

UNIT—IV

4. (A) (i) Test the convergence of :

$$\int_0^\infty \frac{x^2+1}{x^4+1}\,dx.$$

(ii) Prove that an absolutely convergent integral converges. 6

(B) Define Gamma function of n. Prove that $\Gamma(n+1) = n\,\Gamma n$, $n > 0$ and $\Gamma(n+1) = n!$, $n = 1, 2, 3....$ 6

OR

4. (C) Prove the relation $\beta(m, n) = \dfrac{\Gamma m\,\Gamma N}{\Gamma(m+n)}$, $m, n > 0$. 6

(D) Prove that $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$. Hence evaluate

$$\int_0^1 \sqrt{\log \frac{1}{x}}\,dx.$$ 6

UNIT—V

5. (A) Find the value of n for which $\vec{F} = (n-4)\,xi + yj + zk$ is solenoidal. 1½

(B) Find the work done by a force field $\vec{F} = xi - y^2 j$ in moving a particle from (1, 1) to (2, 2) along a straight line y = x. 1½

(C) Find the area of the rectangular region R bounded by the lines x = 0, x = 2, y = 0 and y = 3, using double integral. 1½

(D) Evaluate $\int_0^1 \int_0^2 (x+3)\,dx\,dy$. 1½

(E) Apply Green's theorem to prove that the area enclosed by a simple plane curve C is $\frac{1}{2}\oint_C (x\,dy - y\,dx)$. 1½

(F) Show that :

$$\iint_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \circ \hat{n}\, dS = \frac{4}{3}\pi(a+b+c)$$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$, by divergence theorem. 1½

(G) Test the convergence of :

$$\int_1^\infty \frac{dx}{x^3+5}$$ by comparison test. 1½

(H) Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$. 1½