Bachelor of Science (B.Sc.) Semester—II (C.B.S.) Examination

MATHEMATICS

(Vector Calculus and Improper Integrals)

Compulsory Paper—2

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) If a particle moves along a curve whose parametric equations are :

$x = e^{-t}$, $y = 2 \cos 3t$, $z = 2 \sin 3t$,

where t is the time, then :

(i) Determine its velocity and acceleration at any time t.

(ii) Find the magnitudes of the velocity and acceleration at the time $t = 0$. 6

(B) If $\overline{F} = (3x^2y - z)\hat{i} + (xz^3 + y^4)\overline{j} - 2x^3z^2\overline{k}$, the find :

(i) $\overline{\nabla}(\overline{\nabla} \circ \overline{F})$

(ii) $\overline{\nabla} \times \overline{F}$. 6

OR

(C) Show that $\overline{F} = (2xy + z^3)\overline{i} + x^2\overline{j} + 3xz^2\overline{k}$ is irrotational. Find a scalar potential ϕ such that $\overline{F} = \overline{\nabla}\phi$. 6

(D) If $\overline{A} = (3x^2 + 6y)\overline{i} - 14yz\overline{j} + 20xz^2\overline{k}$. Then evaluate $\int_C \overline{A} \circ d\overline{r}$ from (0, 0, 0) to (1, 1, 1) along the path C : $x = t$, $y = t^2$, $z = t^3$. 6

UNIT—II

2. (A) Evaluate $\iint_R xy\, dxdy$ over the region R in the xy-plane bounded by the circle $x^2 + y^2 = 4$ in the first quadrant. 6

(B) Evaluate $\int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dxdy$ by changing the order of the integration. 6

OR

(C) Evaluate $\iint_R (x^2 + y)\, dxdy$ by changing into polar coordinates, where R is the region $x^2 + y^2 \le 1$. 6

(D) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dxdydz$. 6

UNIT—III

3. (A) Evaluate $\iiint_V x^2 y \, dV$, where V is the closed region bounded by the planes :

$x + y + z = 1,\ x = 0,\ y = 0,\ z = 0.$ 6

(B) Evaluate $\oint_C (y - \sin x)\, dx + \cos x \, dy$ by using Green's theorem in a plane, where C is the triangle in xy-plane with vertices $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 1\right)$. 6

OR

(C) Evaluate $\iint_S (\overline{\nabla} \times \overline{A}) \circ \overline{n} \, dS$, where $\overline{A} = (x^2 + y - 4)\overline{i} + 3xy\,\overline{j} + (2xz + z^2)\,\overline{k}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane. 6

(D) Evaluate $\iint_S \overline{F} \circ \overline{n} \, dS$, where $\overline{F} = 4xz\overline{i} - y^2\overline{j} + yz\,\overline{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ by Divergence theorem. 6

UNIT—IV

4. (A) By quotient test for integrals, test the convergence of :

(i) $\int_0^{\infty} e^{-x^2} dx$

(ii) $\int_1^{\infty} \frac{\log x}{x + a} dx$,

where a is a positive integer. 6

(B) If $\int_a^{\infty} |f(x)|\, dx$ convergens, then prove that $\int_a^{\infty} f(x) dx$ converges. Hence prove that $\int_0^{\infty} \frac{\cos x}{x^2 + 1} dx$ is convergent. 6

OR

(C) Prove that :

(i) $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$,

(ii) $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$. 6

(D) Prove the relation :

$$\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma (m + n)}.$$ 6

Question—V

5. (A) Find $\overline{\nabla}\phi$ if $\phi = \log r$, where $r = |\bar{r}|$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. 1½

(B) Evaluate $\int_C (2x + y^2)\,dx + (3y - 4x)\,dy$ from (0, 0) to (1, 1) along the parabolic path $C : y = x^2$. 1½

(C) Evaluate $\int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta$. 1½

(D) Evaluate $\int_0^1 \int_0^x \int_0^y dz\,dy\,dx$. 1½

(E) Apply Green's theorem to show that area bounded by a simple closed curve C is $\frac{1}{2}\oint_C x\,dy - y\,dx$.

1½

(F) If S is any closed surface enclosing a volume V and $\overline{A} = ax\bar{i} + by\bar{j} + cz\bar{k}$, then prove that $\iint_S \overline{A} \circ \bar{n}\, dS = (a + b + c)V$ by divergence theorem. 1½

(G) Test the convergence of $\int_2^{\infty} \frac{x^2 - 1}{\sqrt{x^6 + 16}}\,dx$ by comparison test. 1½

(H) Evaluate $\beta\left(\frac{1}{2}, \frac{3}{2}\right)$. 1½