NJR/KS/18/3076

Bachelor of Science (B.Sc.) Semester—III (C.B.S.) Examination

MATHEMATICS (Differential Equations and Group Homomorphism)

Paper—II

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) Prove that $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$. Hence deduce, $xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$. 6

(B) Prove that :

(i) $\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left[\frac{n}{x}J_n^2 - \left(\frac{n+1}{n}\right)J_{n+1}^2\right]$,

(ii) $J_0^2 + 2(J_1^2 + J_2^2 + ...) = 1$. 6

OR

(C) Prove that $\int_{-1}^{1} P_m(x) \cdot P_n(x)\, dx = 0$; if $m \neq n$. 6

(D) Prove the recurrence formula :

$nP_n = (2n - 1)x\, P_{n-1} - (n - 1)\, P_{n-2}$, $n \geq 2$. 6

UNIT—II

2. (A) If $L[f(t)] = F(s)$, then prove that :

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s),\ n = 1, 2, 3, ...$$

Hence evaluate $L[t \cdot \sin 2t]$. 6

(B) If $L[f(t)] = F(s)$, then prove that :

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\, ds, \text{ provided } \lim_{t \to 0} \frac{f(t)}{t} \text{ exists.}$$

Hence prove that $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$. 6

OR

(C) If $L[f(t)] = F(s)$, then prove that :

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot F(s), \text{ provided the limit exists.}$$

Verify this result for the function $f(t) = e^{-2t}$. 6

(D) By using convolution theorem, evaluate :

$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right].$$ 6

UNIT—III

3. (A) Solve $y'' + y' - 2y = t$, given that $y(0) = 1$, $y'(0) = 0$. 6

(B) Solve $y'' + ty' - y = 0$, given that $y(0) = 0$, $y'(0) = 2$. 6

OR

(C) Let $u(x, t)$ be a function defined for $t > 0$ and $x \in [a, b]$. Show that :

(i) $L\left[\frac{\partial u}{\partial t}\right] = s\,U - u(x, 0)$, where $U = U(x, s) = L[u(x, t)]$.

(ii) $L\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2 U - s\,u(x, 0) - u_t(x, 0)$, where $u_t(x, 0) = \frac{\partial u}{\partial t}$ at $t = 0$. 6

(D) Find the Fourier sine transform of $e^{-|x|}$ and hence show that :

$$\int_0^\infty \frac{x \cdot \sin mx}{1 + x^2}\, dx = \frac{\pi}{2} e^{-m}, \; m > 0.$$ 6

UNIT—IV

4. (A) If N and M are normal subgroups of a group G, then show that NM is also a normal subgroup of group G. 6

(B) Let G be a group and N be its normal subgroup. Then prove that the set G/N of all cosets is a group with respect to multiplication of cosets. 6

OR

(C) Let G be the additive group of real numbers and G′ be the multiplicative group of positive real numbers. Then show that a mapping $f : G' \to G$ defined by $f(x) = \log_e x$, $\forall\, x \in G$ is an isomorphism of G onto G′. 6

(D) Prove that every quotient group G/H of a given group G is a homomorphic image of the group G and also prove that the kernel of homomorphism is H. 6

QUESTION—V

5. (A) Prove that $\int_a^b J_0(x) J_1(x)\, dx = \frac{1}{2}\{J_0^2(a) - J_0^2(b)\}$. 1½

(B) Prove that $P_2(x) = \frac{1}{2}(3x^2 - 1)$. 1½

(C) Find $L^{-1}\left[\frac{12}{4s - 8}\right]$. 1½

(D) Prove that :

$L[a\, f(t) + b\, g(t)] = a\, L[f(t)] + b\, L[g(t)]$,

where $t > 0$ and a, b are any constants. 1½

(E) Show that $L\left[\frac{\partial u}{\partial x}\right] = \frac{dU}{dx}$, where $U = L[u(x, t)]$. 1½

(F) Find the Fourier sine transform of the function $f(x) = e^{-ax}$. 1½

(G) Let G and G′ be two groups and $f : G \to G'$ be a homomorphism. Then prove that $f(e) = e'$, where e and e′ are identity elements in group G and G′ respectively. 1½

(H) Find all the generators of a cyclic group $G = \{1, -1, i, -i\}$. 1½