NRT/KS/19/2136

Bachelor of Science (B.Sc.) Semester—V Examination

METRIC SPACE, COMPLEX INTEGRATION AND ALGEBRA

Optional Paper—2

(Mathematics)

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve **ALL** the five questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) Define a countable set. If A is the set of all integers, then prove that A is countable. 6

(B) Let X be a metric space with metric d. Show that the function d_1 defined by :

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X. 6

OR

(C) Prove that a set E is open if and only if its complement is closed. 6

(D) Prove that :

(i) Arbitrary intersection of closed sets is closed

(ii) Finite union of closed sets is closed. 6

UNIT—II

2. (A) Define a Cauchy Sequence. Prove that every convergent sequence in a metric space is a Cauchy Sequence. 6

(B) Prove that compact subsets of metric spaces are closed. 6

OR

(C) Suppose $K \subset Y \subset X$. Then prove that K is compact relative to X if and only if K is compact relative to Y. 6

(D) If $\{E_n\}$ is a sequence of non-empty closed sets in a complete metric space X, if $E_{n+1} \subset E_n$, and if $\lim\limits_{n \to \infty} \text{diam } E_n = 0$, then prove that $\bigcap\limits_{n=1}^{\infty} E_n$ consists of exactly one point. 6

UNIT—III

3. (A) If R is cumulative ring and $a \in R$, then show that $aR = \{ar / r \in R\}$ is a two-sided ideal of R. 6

(B) Prove that every finite integral domain is a field. 6

OR

(C) Let $J(\sqrt{2})$ be a ring of real numbers of the form $m + n\sqrt{2}$ (where m and n are integers) under usual addition and multiplication of real numbers. Define $\phi : J(\sqrt{2}) \to J(\sqrt{2})$ by $\phi\,(m + n\sqrt{2}) = m - n\sqrt{2}$. Then prove that :

(i) ϕ is a homomorphism and

(ii) $I(\phi) = \{0\}$. 6

(D) Let U be an ideal of a ring R then prove that R/U is a ring. 6

UNIT—IV

4. (A) Evaluate $\int_0^{1+i} (x - y + ix^2)\, dx$ along the straight line from z = 0 to z = 1 + i. 6

(B) Verify Cauchy's theorem for the function $f(z) = z^3 - iz^2 - 5z + 2i$, if path is a circle given by $|z - 1| = 2$. 6

OR

(C) Using Cauchy integral formula, evaluate $\int_C \frac{(z-1)\, dz}{(z+1)^2 (z-2)}$, where C is the circle $|z - i| = 2$.

6

(D) (i) Find the residue of $\frac{z^3}{z^2 - 1}$ at $z = \infty$.

(ii) Evaluate $\int_C z\, e^{\frac{1}{z}}\, dz$ around the unit circle. 6

QUESTION –V

5. (A) Show that the set of all irrational numbers is uncountable. 1½

(B) If X is a metric space and $E \subset X$, then prove that $E = \overline{E}$ if and only if E is closed. 1½

(C) Define K-cell and explain 1-cell. 1½

(D) Define separated sets. Show that if A = [0, 1] and B = (1, 2) then A and B are not separated. 1½

(E) Show that every field is an integral domain. 1½

(F) If ϕ is a homomorphism of Ring R into R′, then show that $\phi(0) = 0'$, where $0' \in R'$. 1½

(G) Define Rectifiable curve and contour. 1½

(H) Evaluate $\int_0^{1+i} z\, dz$ along the line z = 0 to z = 1 + i. 1½