Bachelor of Science (B.Sc.) Semester-V Examination

METRIC SPACE, COMPLEX INTEGRATION AND ALGEBRA

Optional Paper—2

Mathematics

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) Define a countable set and prove that the set of all sequences whose elements are the digits 0 and 1 is uncountable. 6

(B) In the metric space of real numbers $\mathbb{R}$ with usual metric, show that the function $d(x, y) = \dfrac{|x-y|}{1+|x-y|}$, $\forall$ x, y $\in \mathbb{R}$ is a metric. 6

OR

(C) Prove that the finite intersection of open sets is open. 6

(D) Define interior of a set E. Prove that the interior of E is an open subset of E. 6

UNIT—II

2. (A) Prove that the sequence $\{x_n\}$ of real numbers is a Cauchy sequence if and only if it is convergent in R (a set of real numbers). Hence prove R is a complete metric space. 6

(B) If E is an infinite subset of a compact set K, then prove that E has a limit point in K. 6

OR

(C) Prove that a closed subset of a complete metric space is complete. 6

(D) If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ (n = 1, 2, 3, ...), then prove that $\bigcap_{n=1}^{\infty} I_n$ is not empty. 6

UNIT—III

3. (A) Prove that a commutative ring R is an integral domain if and only if for a, b, c $\in$ R with $a \neq 0$, $ab = ac \Rightarrow b = c$. 6

(B) If U, V are ideals of R, let $U + V = \{u + v \mid u \in U, v \in V\}$, then prove that U + V is also an ideal. 6

OR

(C) If R is a ring with unit element 1 and ϕ is a homomorphism of R onto R' then prove that :

(i) $\phi(0) = 0'$, where 0 and 0' are the zero elements of R and R' respectively

(ii) $\phi(-a) = -\phi(a)\ \forall\ a \in R$ and

(iii) $\phi(1)$ is the unit element of R'. 6

(D) If U and V are ideals of a ring R, then prove that $U \cap V$ is also an ideal of R. 6

UNIT—IV

4. (A) If f(z) is analytic function of z and if f'(z) is continuous at each point within and on a closed contour C, then prove that :

$$\int_C f(z)\, dz = 0 .$$ 6

(B) Evaluate $\int_C \frac{z+4}{z^2+2z+5}$, where C is the circle $|z + 1| = 1$. 6

OR

(C) Evaluate $\int_C \frac{e^z}{z^2(z^2+9)} dz$ by the method of calculus of residues, where C is the circle $|z| = 4$. 6

(D) Prove that :

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} .$$ 6

QUESTION—5

5. (A) Give a counter example to show that an arbitrary intersection of open sets need not be open. 1½

(B) Prove that $\left(\bigcap_\alpha E_\alpha\right)^C = \bigcup_\alpha (E_\alpha^C)$ for the collection of sets E_α. 1½

(C) Show that the sequence $\{x_n\}$, where $x_n = \frac{1}{n}$, $n \in N$ is a Cauchy sequence in R. Here N and R are respectively the sets of natural numbers and real numbers. 1½

(D) Prove by giving an example that the separated sets are disjoint but disjoint sets need not be separated. 1½

(E) Define :

(i) Integral domain

(ii) Division ring. 1½

(F) Prove that if an ideal U of a ring R contains a unit element of R then U = R. 1½

(G) Determine whether the Cauchy's integral theorem is applicable for $\int_C \frac{z^2 + 5z + 6}{z - 2} dz$ if C is the circle $|z| = 1$. 1½

(H) Evaluate the residues of $\frac{z}{(z-1)(z-2)(z-3)}$ at 1, 2, 3. 1½