TKN/KS/16/5913

Bachelor of Science (B.Sc.) (Mathematics) (C.B.S.)
Semester–VI Examination

M_{11} ABSTRACT ALGEBRA

Compulsory Paper—1

Time—Three Hours] [Maximum Marks—60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. 1 to 4 have an alternative. Solve each question in full or its alternative in full.

UNIT–I

1. (A) Show that in a group G, the mapping $T : G \to G$ defined by $T(x) = x^{-1}\ \forall\ x \in G$ is an automorphism of G if and only if G is abelian. 6

(B) For any group G, prove that I(G) is a normal subgroup of A(G), where I(G) is group of inner automorphisms of G and A(G) is group of all automorphisms of G. 6

OR

(C) If G is a finite group and $H \neq G$ is a subgroup of G such that $O(G) \times i(H)!$, then prove that H must contain a nontrivial normal subgroup of G. 6

UNIT–III

3. (A) Let a mapping $T : V_2 \to V_2$ be defined by $T(x, y) = (x', y')$, where $x' = x \cos \theta - y \sin \theta$, $y' = x \sin \theta + y \cos \theta$. Show that T is a linear map. 6

(B) Let $T : U \to V$ be a linear map. Then prove that :

(a) R(T) is a subspace of vector space V.

(b) N(T) is a subspace of vector space U. 6

OR

(C) Let $T : U \to V$ be a nonsingular linear map. The prove that $T^{-1} : V \to U$ is linear, one-one and onto map. 6

(D) Let $T : V_3 \to V_3$ be a linear map defined by

$T(e_1) = e_3$, $T(e_2) = e_1$, $T(e_3) = e_2$.

Prove that $T^2 = T^{-1}$. 6

UNIT–IV

4. (A) Find the matrix of linear map $T : V_3 \to V_3$ defined by :

$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + 3x_2 - \frac{1}{2}x_3, x_1 + x_2 - 2x_3)$ relative to the bases

$B_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$B_2 = \{(1, 1, 0), (1, 2, 3), (-1, 0, 1)\}$. 6

(B) Find the range, kernel, rank and nullity of the matrix :

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$ 6

OR

(C) Define Inner Product Space. Let V be a set of all continuous complex-valued functions on the closed interval [0, 1]. For $f, g \in D$ define

$$f.g = \int_0^1 f(t)\,\overline{g(t)}\,dt$$

show that f.g defines an inner product on V. 6

(D) Using Gram-Schmidt orthogonalization process, orthonormalize the set of linearly independent vectors $\{(1, 0, 1, 1), (-1, 0, -1, 1), (0, -1, 1, 1)\}$ of V_4. 6

Question–V

5. (A) Show that conjugacy relation 'N' on a group G is symmetric. 1½

(B) If S is a finite set of n elements, then state Cayley's theorem. 1½

(C) Prove that the set

$B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of vector space V_3. 1½

(D) Let G be a group and $a \in G$. Define the normalizer $N(a)$ of a in G and prove that $N(a)$ is a subgroup of G. 6

UNIT–II

2. (A) In any vector space V over the field F, prove the following :

   (a) $\alpha 0 = 0$ for every scalar $\alpha \in F$ and $0 \in V$.

   (b) $0u = 0$ for every $u \in V$ and $0 \in F$.

   (c) $\alpha u = 0 \Leftrightarrow \alpha = 0$ or $u = 0$, where $\alpha \in F$ and $u \in V$. 6

(B) If S is a non empty subset of a vector space V over the field F, then prove that [S] is the smallest subspace of V containing S. 6

OR

(C) Let $S = \{(1, 1, 0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}$

   (i) Show that the ordered set S is linearly dependent.

   (ii) Locate one of the vectors that blongs to the span of the previous ones.

   (iii) Find the largest linearly independent subset of S. 6

(D) Prove that in a^n n-dimensional vector space V, any set of n linearly independent vectors is a basis. 6

(D) If U and W are finite dimensional subspaces of a vector space V and $U + W = U \oplus W$, then prove that

$$\dim (U + W) = \dim U + \dim W. \quad 1½$$

(E) Let $T : V_2 \to V_2$ be a linear map defined by

$T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)\}$, show that T is one-one. 1½

(F) Let $T : V_3 \to V_2$ be defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1)$$

and $S : V_2 \to V_2$ be defined by

$S(x_1, x_2) = (x_2, x_1)$. Then determine ST. 1½

(G) Let V be an inner product space. Then for arbitrary vectors u and v in V. Prove that

$$\|u + v\| \leq \|u\| + \|v\|. \quad 1½$$

(H) Show that a matrix $H = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

is orthogonal. 1½