KNT/KW/16/5198

Bachelor of Science (B.Sc.) Semester—VI (C.B.S.) Examination

MATHEMATICS (Abstract Algebra)

Compulsory Paper—1

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the ***five*** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) Define an automorphism of group G. Find whether a mapping $\phi : G \to G$ defined as $\phi(x) = x^2 \ \forall \ x \in G$ is an automorphism, where group $G = (R^+, \cdot)$. 6

(B) Prove that $I(G) \approx G/Z$, where I(G) is the group of inner automorphisms of group G and Z is the centre of group G. 6

OR

(C) If G is a finite group, then prove that :

$$Ca = \frac{O(G)}{O(N(a))}, \text{ where } Ca = O(C(a)).$$ 6

(D) Let Z be the centre of group G and for $a \in G$, N(a) be the normalizer of a in G.

Then prove that :

(i) $a \in Z \Leftrightarrow N(a) = G$

and (ii) if G is finite, then $a \in Z \Leftrightarrow O(N(a)) = O(G)$. 6

UNIT—II

2. (A) Let R^+ be the set of all positive real numbers. Define the operations of addition $\oplus$ and scalar multiplication $\otimes$ as follows :

$u \oplus v = uv \quad \forall \ u, v \in R^+$

and $\alpha \otimes u = u^\alpha \quad \forall \ u \in R^+$ and $\alpha \in F = R$.

Prove that R^+ is a real vector space. 6

(B) If S and T are non empty subsets of a vector space V, then prove that

(i) $S C T \Rightarrow [S] \, C \, [T]$.

(ii) $[S] = S$ if and only if S is a subspace of V.

(iii) $[[S]] = [S]$. 6

OR

(C) Let the set $\{v_1, v_2, ...v_k\}$ be a linearly independent subset of an n-dimensional vector space V. Then prove that we can find vectors $v_{k+1}, v_{k+2}, ...v_n$ in V such that the set $\{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$ is a basis for V. 6

(D) Let $\{(1, 1, 1, 1), (1, 2, 1, 2)\}$ be a linearly independent subset of the vector space V_4. Extend it to the basis for V_4. 6

UNIT—III

3. (A) Let U, V be vector spaces over a field F and $T : U \to V$ be a linear map. Then prove that :

   (a) $T(O_u) = O_v$

   (b) $T(-u) = -T(u), \ \forall \ u \in U$ and

   (c) $T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n), \ \forall \ u_i \in U, \alpha_i \in F, 1 \le i \le n$ and $n \in N$. 6

   (B) Let $T : V_4 \to V_3$ be a linear map defined by $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$. Find range, rank, kernel and nullity of T and verify Rank-Nullity theorem. 6

OR

   (C) Let $T : U \to V$ be a linear map and U a finite-dimensional vector space. Then prove that dim R(T) + dim N(T) = dim U. 6

   (D) Prove that the linear map $T : V_3 \to V_3$ defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$ is nonsingular and find tis inverse. 6

UNIT—IV

4. (A) Let a linear map $T : P_3 \to P_2$ be defined by $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_3 + (\alpha_2 + \alpha_3)x + (\alpha_0 + \alpha_1)x^2$. Then determine matrix of T relative to the bases $B_1 = \{1, (x - 1), (x - 1)^2, (x - 1)^3\}$ and $B_2 = \{1, x, x^2\}$. 6

   (B) Prove that the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix}$ is nonsingular and find its inverse. 6

OR

   (C) In an inner product space V_i prove that :

   (i) $\| u + v \| \le \| u \| + \| v \| \ \forall \ u, v \in V$

   (ii) Any orthogonal set of no zero vectors is linearly independent. 6

   (D) Find the orthonormal basis of $P_2 [-1, 1]$ starting from the basis $\{1, x, x^2\}$ using the inner product defined by $f \cdot g = \int_{-1}^{1} f(x) \cdot g(x)dx$. 6

UNIT—V

5. (A) Show that conjugacy relation '~' on group G is reflexive. 1½

(B) Show that I(G) = {I} for an abelian group G, where I(G) is the set of inner automorphisms of G. 1½

(C) Let $S = \{(x_1, x_2, x_3) \in V_3 / x_2 + x_3 = x_1\}$. Prove that S is a subspace of V_3. 1½

(D) Is the sum x-axis + y-axis in V_3 a direct sum ? 1½

(E) Find whether a mapping $T : V_2 \to V_2$ defined by $T(x, y) = (x + 1, y + 2)$ $\forall$ $(x, y) \in V_2$ is a linear map. 1½

(F) If U and V are finite dimensional vector spaces such that dimU = dimV. Then prove that a linear map $T : U \to V$ is one-one if and only if it is onto. 1½

(G) Show that the matrix $U = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is unitary. 1½

(H) In an inner product space V, prove that $u \cdot (\alpha v) = \bar{\alpha}\,(u \cdot v)$, $\forall$ $u, v \in V$ and $\alpha \in F$. 1½